**Poisson equations of motion**

Just when you thought we were done, there is another way to reformulate these equations. Again, this is important when making the leap from classical mechanics to quantum mechanics. Consider the following. Any general function f(qi, pi, t), including these quantities themselves will evolve according to…



Defining the Poisson bracket of two functions A(qi,pi) and B(qi,pi) as:



we can write this as:



So the time evolution of a quantity is determined by its commutation relation with H. In particular, we can write,



As you can verify, this will reduce to Hamilton’s equations. The Poisson bracket provides a nice way to check whether or not a quantity Q(qi,pi) is conserved or not. If it is conserved, then its time-derivative ought to be 0, which means that it ought to ‘commute’ with H. This is a particularly nice feature in that it’s easy to check out whether some quantity is time-dependent in the Poisson bracket formalism. It’s a bit harder to work this out in the Lagrangian formulation.

**Example**

What is ? What is ? What is  where f(q,p)?

So it is:



and,



and,



These are just like the QM counterparts – as you will see.

**Example**

What is [Li, Lj], where Li = (**r**×**p**)i = εijkrjpk (implicit summation over repeated indices)? Well,



And now we’ll observe:



So we find this can be written as:



**Example**

In the EM Hamiltonian, what is the commutation relation [x,p]?



b/c remember p is the canonical momentum.

**Example**

Consider the Hamiltonian:



Let’s verify that total momentum P = p1 + p2 is conserved, as is was demonstrated via Noether’s theorem. So



So it is a conserved quantity.

**Example**

Consider a Hamiltonian:



Clearly momentum is not conserved. Neither is angular momentum. What if we had a two-atom crystal-like Hamiltonian, where atoms’ equilibrium positions are R1,2…



…we still don’t have conservation of momentum or angular momentum, as could verify. But if we translate the crystal as a whole, i.e., moving R1 and R2 along with x1 and x2, then we seem to, because the Hamiltonian would be invariant, and so by Noether’s theorem we would have a conservation law. But I think this is kind of an illusion. Because there is no momentum associated with ‘particles’ at R1, and R2. So unless the coordinates we’re translating are associated with a particle of some sort, or constitute a ‘dynamical’ variable, the symmetry is kind of false, and doesn’t imply a conservation law strictly speaking. Of course if we attached to particles to a rod via springs or something, then the full Hamiltonian/Lagrangian would include the rod, and so the symmetry could be taken advantage of.

**Solution to Poisson’s equation**

We can ‘solve’ this equation, at least in a Taylor series fashion. Let’s say f is not explicitly dependent on time. Then,



And we’ll say initially f(0) = f0 = f(qi(0),pj(0)). Then at some later time, t, we have:



and we can solve this iteratively by plugging the RHS into f(t’).



and we could do it again and we’ll end up with:



If we truncate our expansion by putting f(t´´´) = f0, then we’d have:



Since f0 is independent of time, we can integrate to get:



and if we had continued the recursive expansion indefinitely, we would’ve gotten,



where in the last term there are n Poisson brackets all total. This can be put in suggestive form,



which is *defined* as the expansion above.

**Generators**

Looking at the equation:



and assuming ∂f/∂t = 0, we can write:



So we might call H the generator of time translation. Let’s go back to a previous result:



So we have:



Stands to reason then, that we could write:



and we could say that **p** is the generator of spatial translation. It’s also the case that we can write:



We might recognize this as:



where the derivatives are w/r to the azimuthal angle φ. So to evaluate we’d have to put both r and p in spherical coordinates. So we can write:



and in general we may surmise,



where φj is angle about the jth axis. Note how in this case we see that both r and p are rotating by infinitesimal angle dφ. And we may surmise that finite rotations, φ, about are given by:



where r´ and p´ are related to r and p by the rotation matrix r´i = Rij(φ)rj and p´i = Rij(φ)pj [implici summation]. So we can say that **L** is the generator of rotations. So altogether, we have:

